M1 INTERMEDIATE ECONOMETRICS

Linear regression: Asymptotics

Koen Jochmans   François Poinas

2025 — 2026

Asymptotics for linear regression

We have

$$Y = X'\beta + e, \qquad \mathbb{E}(Xe) = 0.$$

Want to know the sampling behavior of

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{Y})$$

as $n \to \infty$.

Under moment conditions, the least-squares estimator is consistent and asymptotically normal.

The variance of the limit distribution, the asymptotic variance, can be estimated consistently.

Then treat it as approximately normal when n is not too small.

Consistency

We say that $\hat{\beta}$ is consistent for β if, for any $\varepsilon > 0$, it holds that

$$\mathbb{P}(\|\hat{\beta} - \beta\| > \varepsilon) \to 0$$

as $n \to \infty$.

The probability that $\hat{\beta}$ lies beyond a distance ε from β goes to zero as the sample size grows large.

In our context this only requires that $\mathbb{E}(\|X\|^2) < +\infty$ and that $\mathbb{E}(XX')$ is full rank.

To begin,

$$\hat{\beta} - \beta = (\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{e}) = \left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}\left(\frac{\boldsymbol{X}'\boldsymbol{e}}{n}\right)$$

is the ratio of two sample averages.

The numerator satisfies

$$\frac{\boldsymbol{X}'\boldsymbol{e}}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i e_i \underset{p}{\to} \mathbb{E}(Xe) = 0.$$

For the denominator,

$$\frac{\boldsymbol{X}'\boldsymbol{X}}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i X_i' \underset{p}{\to} \mathbb{E}(XX'), \qquad \left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1} \underset{p}{\to} \mathbb{E}(XX')^{-1}.$$

The product of the two converges to the product of their respective probability limits:

$$\hat{\beta} - \beta \underset{p}{\to} \mathbb{E}(XX')^{-1}\,0 = 0.$$

The rate at which $\|\hat{\beta} - \beta\|$ approaches 0 is $n^{-1/2}$.

Scaling up by $n^{1/2}$, will prevent it to collapse, instead staying random as $n \to \infty$.

We have

$$\sqrt{n}(\hat{\beta} - \beta) \underset{d}{\to} N(0, V_\beta), \qquad V_\beta = Q^{-1} \Omega \, Q^{-1},$$

for

$$Q = \mathbb{E}(XX'), \qquad \Omega = \mathrm{var}(Xe) = \mathbb{E}(e^2 XX'),$$

if we additionally assume that $\mathbb{E}(\|X\|^4) < +\infty$ and that $\mathbb{E}(e^4) < +\infty$.

The central limit theorem gives

$$\left(\frac{\boldsymbol{X}'\boldsymbol{e}}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i e_i \underset{d}{\rightarrow} N(0, \mathit{\Omega})$$

while, from before,

$$\left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \underset{p}{\rightarrow} \mathbb{E}(XX')^{-1} = Q^{-1}.$$

So,

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}\left(\frac{\boldsymbol{X}'\boldsymbol{e}}{\sqrt{n}}\right) \underset{d}{\rightarrow} N(0, Q^{-1}\mathit{\Omega}\, Q^{-1})$$

follows.

Variance estimation

We estimate V_β by

$$\hat{V}_\beta = \hat{Q}^{-1} \hat{\Omega} \hat{Q}^{-1}$$

with

$$\hat{Q} = \frac{1}{n} \sum_{i=1}^{n} X_i X_i', \qquad \hat{\Omega} = \frac{1}{n} \sum_{i=1}^{n} \hat{e}_i^2 X_i X_i^2,$$

where

$$\hat{e}_i = Y_i - X_i' \hat{\beta}$$

are the least-squares residuals.

Can show that $\hat{V}_\beta \underset{p}{\to} V_\beta$.

Then use the approximation

$$\hat{\beta} \underset{a}{\sim} N(\beta, \hat{V}_\beta / n).$$

When regressor errors are homoskedastic,

$$\Omega = \mathbb{E}(e^2 X X') = \mathbb{E}(\mathbb{E}(e^2|X) X X') = \mathbb{E}(e^2)\,\mathbb{E}(X X')$$

so that $V_\beta = \sigma^2\, Q^{-1}$.

We estimate the asymptotic variance as

$$\hat{\sigma}^2 \hat{Q}^{-1} = n\,\hat{\sigma}^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}$$

for

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{i=1}^{n} \hat{e}_i^2 = (\hat{\boldsymbol{e}}'\hat{\boldsymbol{e}})/(n-k),$$

which is again consistent.

We then use the approximation

$$\hat{\beta} \underset{a}{\sim} N(\beta, \hat{\sigma}^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}).$$

Compare this to the classical linear regression setup.